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Viscous attenuation of mean drift in water waves

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The initial-value problem for slightly viscous, two-dimensional, spatially periodic waves is examined. Matched asymptotic expansions in space for small wave amplitude a and multiple scales in time allow the boundary layers and viscous attenuation to be described. The bottom and surface boundary layers of thickness δ are equivalent to those of Longuet-Higgins except that wave attenuation is included. For progressive waves one solution for the interior motion independent of the magnitude of δ/a is an attenuating version of the conduction solution of Longuet-Higgins, but with modified structure, the $O(a^2)$ vorticity at the boundaries ultimately diffusing into the entire field. There are certain critical depths for which there is secular behaviour and these do not correspond to quasi-steady flows. Other solutions may be possible. For standing waves the interior flow depends on the magnitude of the steady-drift Reynolds number $R_s \propto (a/\delta)^2$ introduced by Stuart. When $R_s \ll 1$, the interior is viscous with an $O(a^2)$ vorticity ultimately diffusing into the entire field. When $R_s \gg 1$ there is a double-boundary-layer structure on the bottom and on the surface. Within the outer layers, the $O(a^2)$ steady drift decays to the potential flow interior. A direct analogy with the flow structure on a circular cylinder oscillating along its diameter is introduced and pursued. Finally, all of the above fields are converted to Lagrangian fields so that mass-transport profiles can be obtained. Comparisons are made with previous theoretical and experimental work.

1. Introduction

It was shown by Stokes (1847) for an inviscid progressive water wave of small amplitude a that the particles of fluid possess, apart from their closed orbital motions, steady-drift velocities $O(a^2)$ in the direction of wave propagation. Since then, various experiments (e.g. Caligny 1878; Bagnold 1947) have confirmed the existence of non-zero mean velocities but the profiles are not predicted by the Stokes theory.

Longuet-Higgins (1953) showed that even for the very high Reynolds numbers encountered in experiment viscous effects significantly modify the drift (also see Harrison 1909). Longuet-Higgins posed the existence of boundary layers of thickness δ on the channel bottom and at the free surface. Just outside the bottom boundary layer there is a constant *mean drift* $O(a^2)$ while outside the surface boundary layer there is a *mean drift gradient* $O(a^2)$. For the interior motion (outside both boundary layers) Longuet-Higgins suggested two possibilities. The interior motion is called 'conduction' if $a \ll \delta$ while it is called 'convection' if $a \gg \delta$. In the latter case only an *ad hoc* 'solution'

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is obtained. Subsequent workers, most notably Dore (1971, 1974, 1975), have followed this approach and extended the range of application to other systems.

There are certain deficiencies in the Longuet-Higgins analysis. Huang (1970) points out that the (Lagrangian) mass-transport velocity on the surface becomes unbounded as the depth d approaches infinity. The Longuet-Higgins analysis reveals the role played by Reynolds stresses in producing drift in a slightly viscous fluid. However, a second effect of viscosity, seemingly of comparable importance, is neglected. Unless the external world supplies energy (e.g. through a wave maker), the whole wave field attenuates. Hunt & Massoud (1962) found that an $O(a^2)$ shear stress on the surface is required to maintain the field.

Sleath (1973) attempts to assess the spatial decay of the drift in progressive waves for shallow water and claims to show non-uniqueness of the drift solutions. However, various approximations make the results uncertain.

The most complete set of experimental data on mass transport in *progressive waves* is that of Russell & Osorio (1957). Since a typical value of δ/a is 0·03, the Longuet-Higgins (1953) 'conduction solution', which is proposed for $\delta/a \gg 1$, should not apply. However, Russell & Osorio (1957) find that the mass transport correlates best with the Stokes inviscid theory when the water depth is large while for shallower water the conduction solution is the best available comparison. However, they conclude that apart from the profile at the bottom of the channel, no theory satisfactorily predicts mass transport. Ünlüata & Mei (1970) concluded that the Longuet-Higgins theory predicts the drift better than any other available theory.

Experimental measurements of the mass-transport velocity in the bottom boundary layer in *standing waves* were carried out using dye-streak and solid-particle methods by Noda (1968), and the experimental evidence is in rough qualitative agreement with the theory of Longuet-Higgins.

The present study is an analysis of the initial-value problem of two-dimensional water waves that are spatially periodic. Both *progressive and standing waves* are considered. As time passes, the viscous attenuation in time is examined and the full Eulerian wave field is calculated by matched asymptotic expansions in space for small wave amplitude a and by multiple scales in time. The slow time $\tilde{t}$, which was obtained by Lamb (1932, p. 627) for linearized theory and measures the viscous decay, is derived from the condition of zero normal stress at the surface (surface tension is neglected). This temporal decay can be converted to spatial decay with distance from a wave maker through the group velocity (e.g. see Phillips 1966).

To leading order, the wave attenuation does not modify the velocity profiles in the boundary layers, apart from an exponentially decaying envelope that depends on $\tilde{t}$. The slow decay of the field *strongly modifies the structure of the interior* even apart from the slowly decaying envelope. The Reynolds number R_s characteristic of the steady drift is introduced. For *progressive waves*, the vorticity will diffuse inwards from the boundary layers. A quasi-steady, fully viscous solution, a modified version of the Longuet-Higgins 'conduction' solution, then exists for all R_s, i.e. all a/δ for almost all depths d. However, there are certain depths at which no quasi-steady state will be reached. For large R_s other solutions may exist as well. For *standing waves*, when $R_s \gg 1$, Stuart (1966) argues in a different context that the steady drift is confined to an outer boundary layer within which the steady drift decays to the potential solution. The outer-layer problem for standing waves displays a cellular structure and is shown to be equivalent to the outer layer on a circular cylinder oscillating along its diameter (Davidson & Riley 1972). The analogy is pursued in detail. For $R_s \ll 1$, the interior motion is fully viscous. The analysis of the interior flows is the principal feature of the present work.

Finally, all the Eulerian drift fields are converted to Lagrangian mass-transport fields and compared with previous work, both theoretical and experimental.

2. Formulation

Let us consider two-dimensional gravity waves in a channel of mean depth d. The liquid has constant density ρ, viscosity μ and $\nu = \mu/\rho$. We define a system of Cartesian co-ordinates with the x axis horizontal and the z axis anti-parallel to gravity g. The origin lies at the mean surface position. An x-periodic wave of amplitude a, wave-number k and phase speed c is assumed to evolve in time t from a harmonic initial form. We wish to follow the time evolution and decay.

We non-dimensionalize the governing system of equations with the following scales:

$$\begin{aligned} &\text{length} && \rightarrow 1/k, \\ &\text{speed} && \rightarrow c_0 = (g \tanh kd/k)^{\frac{1}{2}}, \\ &\text{pressure} && \rightarrow \rho c_0^2, \\ &\text{time} && \rightarrow (kc_0)^{-1}. \end{aligned}$$

The flow is governed by the non-dimensional Navier–Stokes and continuity equations, and the following non-dimensional numbers emerge:

$$\alpha = ka, \quad \beta = k^2d^2, \quad R = c_0/k\nu.$$

The flow Reynolds number R is considered large. The boundary conditions for these viscous waves are zero cross-flow and no slip at the bottom $z = -\sqrt{\beta}$. On the displaced level $z = h(x,t)$ there is the kinematic condition, zero shear stress and, if we neglect surface tension, zero normal stress. At time $t = 0$, the vorticity is zero and

$$h(x,0) = \alpha \cos x + O(\alpha^2). \tag{2.1}$$

The existence of viscosity in the liquid forces the motion to decay with a time scale $\bar{t}$. We shall determine $\bar{t}$ in § 6 from the condition of zero normal stress; it turns out that

$$\bar{t} = 2t/R. \tag{2.2}$$

All dependent variables are hence functions of both t and $\bar{t}$ through a multiple-time-scale procedure (Nayfeh 1973, chap. 6).

We seek solutions of the governing system for small values of α as follows:

$$h = \alpha h_1 + \alpha^2 h_2 + O(\alpha^3), \tag{2.2a}$$

$$(u, w) = \alpha(u_1, w_1) + \alpha^2(u_2, w_2) + O(\alpha^3), \tag{2.2b}$$

$$p = z/\tanh\sqrt{\beta} + \alpha p_1 + \alpha^2 p_2 + O(\alpha^3), \tag{2.2c}$$

$$c = 1 + O(\alpha^2). \tag{2.2d}$$

The $O(\alpha)$ correction to c is known to vanish (Wehausen & Laitone 1960).

3. Potential theory

Since we consider that $R \gg 1$, we first examine the initially irrotational flow. The solutions (Wehausen & Laitone 1960) for *progressive waves* at order α are as follows:

$$h_1 = A(\bar{t})\cos(x-t), \tag{3.1a}$$

$$\hat{u}_1 = A(\bar{t})\frac{\cosh(z+\sqrt{\beta})}{\sinh\sqrt{\beta}}\cos(x-t), \tag{3.1b}$$

$$\hat{w}_1 = A(\bar{t})\frac{\sinh(z+\sqrt{\beta})}{\sinh\sqrt{\beta}}\sin(x-t), \tag{3.1c}$$

$$\hat{p}_1 = A(\bar{t})\frac{\cosh(z+\sqrt{\beta})}{\sinh\sqrt{\beta}}\cos(x-t). \tag{3.1d}$$

At order α^2 we have

$$h_2 = \tfrac{1}{2}A^2(\bar{t})D \quad \cos 2(x-t), \tag{3.2a}$$

$$\hat{u}_2 = 2A^2(\bar{t})B\frac{\cosh 2(z+\sqrt{\beta})}{\sinh 2\sqrt{\beta}}\cos 2(x-t), \tag{3.2b}$$

$$\hat{w}_2 = 2A^2(\bar{t})B\frac{\sinh 2(z+\sqrt{\beta})}{\sinh 2\sqrt{\beta}}\sin 2(x-t), \tag{3.2c}$$

where

$$D = \frac{2+\cosh 2\sqrt{\beta}}{\tanh^2\sqrt{\beta}\sinh 2\sqrt{\beta}}, \quad B = \frac{3(\coth\sqrt{\beta}-\tanh\sqrt{\beta})}{4\tanh^2\sqrt{\beta}}.$$

The solutions for *standing waves* at order α are as follows:

$$h_1 = A(\bar{t})\cos x\cos t, \tag{3.3a}$$

$$\hat{u}_1 = -A(\bar{t})\frac{\cosh(z+\sqrt{\beta})}{\sinh\sqrt{\beta}}\sin x\sin t, \tag{3.3b}$$

$$\hat{w}_1 = A(\bar{t})\frac{\sinh(z+\sqrt{\beta})}{\sinh\sqrt{\beta}}\cos x\sin t, \tag{3.3c}$$

$$\hat{p}_1 = -A(\bar{t})\frac{\cosh(z+\sqrt{\beta})}{\sinh\sqrt{\beta}}\cos x\cos t. \tag{3.3d}$$

At order α^2 we have

$$h_2 = \tfrac{1}{2}A^2(\bar{t})D\cos 2x\cos 2t, \tag{3.4a}$$

$$\hat{u}_2 = 2A^2(\bar{t})B\frac{\cosh 2(z+\sqrt{\beta})}{\sinh 2\sqrt{\beta}}\sin 2x\sin 2t, \tag{3.4b}$$

$$\hat{w}_2 = 2A^2(\bar{t})B\frac{\sinh 2(z+\sqrt{\beta})}{\sinh 2\sqrt{\beta}}\cos 2x\sin 2t. \tag{3.4c}$$

Higher-order terms are obtainable directly. We shall *not* consider that β is small, but allow β to be arbitrary.

The amplitude A, which charts the viscous decay of the fields with slow time $\bar{t}$, is at this stage unknown apart from its initial value, given by condition (2.1), i.e.

$$A(0) = 1. \tag{3.5}$$

4. The boundary layer at the bottom

The irrotational flow solutions of § 3 slip over the bottom. In order to enforce the no-slip condition, we insert a boundary layer of thickness $\delta = (2\nu/\omega)^{\frac{1}{2}}$, where $\omega = kc_0$. The result is a modified Stokes layer which in non-dimensional form has a stretched normal co-ordinate

$$\eta = (z+\sqrt{\beta})/\Delta \quad \text{for} \quad R \to \infty, \tag{4.1a}$$

where $\Delta \equiv k\delta = (2/R)^{\frac{1}{2}}$. We scale the normal velocity on Δ as well:

$$w = \Delta W \quad \text{for} \quad R \to \infty, \tag{4.1b}$$

so that $W = O(\alpha)$ in the boundary layer. The boundary-layer equations take the form

$$u_t + uu_x + Wu_\eta = -p_x + \tfrac{1}{2}u_{\eta\eta}, \tag{4.2a}$$

$$p_\eta = 0, \quad u_x + W_\eta = 0. \tag{4.2b, c}$$

The boundary conditions are

$$u = W = 0 \quad \text{on} \quad \eta = 0 \tag{4.2d}$$

and the matching condition is

$$u \to \hat{u}(x, -\sqrt{\beta}) \quad \text{as} \quad \eta \to \infty, \tag{4.2e}$$

where the $\hat{u}$ is the potential solution of § 3. The pressure is initially given by

$$\hat{p}(x, -\sqrt{\beta}, 0).$$

We again represent the solutions in powers of α as in (2.2).

For *progressive waves*, at order α we have

$$u_1 = \frac{A(\tilde{t})}{\sinh\sqrt{\beta}}\{\cos(x-t) - e^{-\eta}\cos(x-t+\eta)\}, \tag{4.3a}$$

$$w_1 = \frac{A(\tilde{t})}{\sinh\sqrt{\beta}}\left\{\eta\sin(x-t) + \frac{e^{-\eta}}{2}\cos(x-t+\eta) + \frac{e^{-\eta}}{2}\sin(x-t+\eta) - \tfrac{1}{2}\cos(x-t) - \tfrac{1}{2}\sin(x-t)\right\}, \tag{4.3b}$$

$$p_1 = \frac{A(\tilde{t})}{\sinh\sqrt{\beta}}\cos(x-t). \tag{4.3c}$$

At order α^2 there are two types of term: those having dependence like $e^{2i(x-t)}$ and those independent of $x-t$. The solutions corresponding to the former are easily obtainable. Those corresponding to the latter are the *steady-drift* contributions that arise from the order-α terms through the generation of Reynolds stresses. These are governed by the following system:

$$\overline{u_1 u_{1x}} + \overline{W_1 u_{1\eta}} = \tfrac{1}{2}\bar{u}_{2\eta\eta}, \tag{4.4a}$$

where the bars indicate the t average over one cycle. There is no pressure contribution since the pressure gradient has zero mean. Note that to this order there is no local time derivative as large as the viscous force. The appropriate boundary conditions on $\bar{u}_2$ are

$$\bar{u}_2 = 0 \quad \text{on} \quad \eta = 0, \tag{4.4b}$$

$$\bar{u}_2 \to 0 \quad \text{as} \quad \eta \to \infty. \tag{4.4c}$$

The solution of (4.4a, b) is

$$\bar{u}_2 = \frac{A^2(\tilde{t})}{\sinh^2\sqrt{\beta}}\left\{\frac{3}{4} - \frac{\eta+2}{2}e^{-\eta}\cos\eta + \tfrac{1}{4}e^{-2\eta} - \frac{\eta-1}{2}e^{-\eta}\sin\eta\right\}. \qquad (4.5a)$$

However, as $\eta \to \infty$ in (4.5a),

$$\bar{u}_2 \to 3A^2(\tilde{t})/4\sinh^2\sqrt{\beta}, \qquad (4.5b)$$

which does not satisfy the matching condition (4.4c). Solution (4.5) agrees with that in Phillips (1966, p. 42).

For *standing waves* at order α we have

$$u_1 = -\frac{A(\tilde{t})\sin x}{\sinh\sqrt{\beta}}\{\sin t - e^{-\eta}\sin(t-\eta)\}, \qquad (4.6a)$$

$$W_1 = \frac{A(\tilde{t})\cos x}{\sinh\sqrt{\beta}}\left\{\eta\sin t + \frac{e^{-\eta}}{2}\sin(t-\eta) - \frac{e^{-\eta}}{2}\cos(t-\eta) + \tfrac{1}{2}\cos t - \tfrac{1}{2}\sin t\right\}, \qquad (4.6b)$$

$$p_1 = -\frac{A(\tilde{t})\cos x}{\sinh\sqrt{\beta}}\cos t. \qquad (4.6c)$$

At order α^2, we again examine only the time-independent part of the solution. Again (4.4a) governs the steady drift while (4.4b, c) give the boundary conditions. The solution of (4.4a, b) is

$$\bar{u}_2 = \frac{A^2(\tilde{t})\sin 2x}{8\sinh^2\sqrt{\beta}}\{-3 + e^{-2\eta} + (8+2\eta)e^{-\eta}\sin\eta + 2(1-\eta)e^{-\eta}\cos\eta\}. \qquad (4.7a)$$

However, as $\eta \to \infty$ in (4.7a)

$$\bar{u}_2 \to -3A^2(\tilde{t})\sin 2x/8\sinh^2\sqrt{\beta}. \qquad (4.7b)$$

Solution (4.7) was first obtained by Noda (1968).

Note that both for progressive [equations (4.5)] and for standing waves [equations (4.7)] the slow time variation enters only through the envelope function A.

5. The boundary layer at the surface

The irrotational flow solutions of § 3 have non-zero shear stress at the top. In order to enforce the condition of zero shear stress, we insert a boundary layer of thickness δ. The result is a layer of modified Stokes type.

Since the boundary-layer thickness may be small compared with the wave amplitude, it is hardly satisfactory to apply the surface boundary condition at the mean water level $z = 0$. The alternative (Longuet-Higgins 1953) is to use a system of curvilinear co-ordinates (s, n, t) in which the free surface is a co-ordinate line. Let κ be the curvature of the surface, and let s and n be co-ordinates along and perpendicular to the surface. The elements of length along the surface and the normal are $(1+\kappa n)\,ds$ and dn respectively. Let $v^{(s)}$ and $v^{(n)}$ denote the corresponding components of velocity. We scale the normal co-ordinate and normal velocity on Δ:

$$\eta = n/\Delta, \quad v^{(n)} = \Delta V^{(n)} \quad \text{for} \quad R \to \infty. \qquad (5.1)$$

The boundary-layer equations then become

$$v_t^{(s)} + \frac{1}{1+\kappa n} v^{(s)} v_s^{(s)} + V^{(n)} v_\eta^{(s)} = -\frac{p_s}{1+\kappa n} + \tfrac{1}{2} v_{\eta\eta}^{(s)}, \tag{5.2a}$$

$$v_t^{(n)} - \frac{\kappa}{1+\kappa n} (v^{(s)})^2 = -\Delta^{-1} p_\eta, \tag{5.2b}$$

$$v_s^{(s)} + [(1+\kappa n)\, V^{(n)}]_\eta = 0. \tag{5.2c}$$

The boundary conditions are

$$V^{(n)} = 0, \quad \Delta^{-1} v_\eta^{(s)} - \kappa v^{(s)} = 0 \quad \text{on} \quad \eta = 0 \tag{5.2d}$$

and the matching condition is

$$v^{(s)} \to \hat{u}(x, h) \quad \text{as} \quad \eta \to -\infty. \tag{5.2e}$$

The solutions are expressible in the form of an asymptotic series in the small parameters α and Δ:

$$(v^{(s)}, V^{(n)}) = \alpha(v_1^{(s)}, V_1^{(n)}) + \alpha\Delta\{(v_2^{(s)}, V_2^{(n)}) + \alpha(v_3^{(s)}, V_3^{(n)}) + O(\alpha^2)\}, \tag{5.3a}$$

$$p = \alpha p_1 + \alpha\Delta\{p_2 + \alpha p_3 + O(\alpha^2)\}. \tag{5.3b}$$

The pressure field is given by potential theory. The curvature κ, which is positive when the surface is concave downwards, is written as

$$\kappa = \alpha\kappa_1 + O(\alpha^2). \tag{5.4}$$

For *progressive waves*, we can choose a reference frame moving with the wave. In this frame, the motion is steady apart from the slow attenuation. In the surface layer, then, let $U = -c + u$, where $|c| \gg |u|$. At $O(\alpha)$ we have

$$v_1^{(s)} = A(\tilde{t}) \coth \sqrt{\beta} \cos s, \tag{5.5a}$$

$$V_1^{(n)} = A(\tilde{t})\, \eta \coth \sqrt{\beta} \sin s, \tag{5.5b}$$

$$p_1 = A(\tilde{t}) \coth \sqrt{\beta} \cos s. \tag{5.5c}$$

This will be the same as the irrotational solution at the surface, and so there is no viscous perturbation to the first order. At $O(\alpha\Delta)$ we have

$$v_2^{(s)} = A(\tilde{t})\{\eta \cos s + e^\eta \sin(s-\eta) - e^\eta \cos(s-\eta)\}, \tag{5.6a}$$

$$V_2^{(n)} = A(\tilde{t})\{\tfrac{1}{2}\eta^2 \sin s + \cos s - e^\eta \cos(s-\eta)\}, \tag{5.6b}$$

$$p_2 = A(\tilde{t})\, \eta \cos s. \tag{5.6c}$$

At $O(\alpha^2\Delta)$, there are two types of term. We are only interested in the s-independent part and this arises as a consequence of the Reynolds stresses. The resulting steady velocity gradient is determined by the following system:

$$\overline{v_2^{(s)} v_{1s}^{(s)}} + \overline{v_1^{(s)} v_{2s}^{(s)}} + \overline{V_1^{(n)} v_{2\eta}^{(s)}} = \tfrac{1}{2} \overline{v_{3\eta\eta}^{(s)}}, \tag{5.7a}$$

where the bars indicate the s average over one wavelength. The appropriate boundary conditions on $\overline{v_{3\eta}^{(s)}}$ are

$$\overline{v_{3\eta}^{(s)}} = \overline{\kappa_1 u_1} \quad \text{on} \quad \eta = 0, \tag{5.7b}$$

$$v_{3\eta}^{(s)} \to 0 \quad \text{as} \quad \eta \to -\infty. \tag{5.7c}$$

The solution of (5.7a,b) is

$$\overline{v_{3\eta}^{(s)}} = A^2(\tilde{t})\coth\sqrt{\beta}\{\tfrac{3}{2} - \eta e^{\eta}\sin\eta + (\eta-1)\,e^{\eta}\cos\eta\}. \qquad (5.8a)$$

However, as $\eta \to -\infty$ in (5.8a),

$$\overline{v_{3\eta}^{(s)}} \to \tfrac{3}{2}A^2(\tilde{t})\coth\sqrt{\beta}, \qquad (5.8b)$$

which does not satisfy the matching condition (5.7c). In contrast to the constant drift velocity in the bottom boundary layer, there is a mean velocity *gradient* induced just outside the surface boundary layer.

The co-ordinate transformation from (x,z) to (s,n) can be written in powers of α as follows:

$$n = z - \alpha\cos x + O(a^2), \qquad (5.9)$$
$$s = x - \alpha z\sin x + O(a^2).$$

At the edge of the boundary layer, then, the interior flow sees a non-zero velocity gradient $\partial\bar{v}_3^{(s)}/\partial n$ [equation (5.8b)], which can be converted to a value of $\partial\bar{u}_2/\partial z$ at $z = 0$ by using the fact that α is small. The result is

$$\partial\bar{u}_2(x,0)/\partial z = 2A^2(\tilde{t})\coth\sqrt{\beta}. \qquad (5.10)$$

For *standing waves*, the nodes are fixed but the free surface is a moving boundary. We adopt the intrinsic co-ordinates (s,n,t) introduced by Longuet-Higgins (1953). At order α, we have

$$v_1^{(s)} = -A(\tilde{t})\coth\sqrt{\beta}\sin s\,\sin t, \qquad (5.11a)$$

$$V_1^{(n)} = A(\tilde{t})\,\eta\coth\sqrt{\beta}\cos s\sin t, \qquad (5.11b)$$

$$p_1 = -A(\tilde{t})\coth\sqrt{\beta}\cos s\cos t. \qquad (5.11c)$$

At order $\alpha\Delta$, we have

$$v_2^{(s)} = A(\tilde{t})\sin s\{-\eta\sin t + e^{\eta}[\sin(\eta+t) - \cos(\eta+t)]\}, \qquad (5.12a)$$

$$V_2^{(n)} = A(\tilde{t})\cos s\{e^{\eta}\cos(\eta+t) - \cos t\}, \qquad (5.12b)$$

$$p_2 = -A(\tilde{t})\,\eta\cos s\cos t. \qquad (5.12c)$$

At order $\alpha^2\Delta$, we again examine only the time-independent part of the solution. Again (5.7a) governs the steady velocity gradient while (5.7b,c) give the boundary conditions. The solution of (5.7a,b) is

$$\overline{v_{3\eta}^{(s)}} = \tfrac{1}{2}A^2(\tilde{t})\coth\sqrt{\beta}\sin 2s\{\eta e^{\eta}\cos\eta + (\eta-3)\,e^{\eta}\sin\eta\}. \qquad (5.13a)$$

As $\eta \to -\infty$ in (5.13a) $\qquad v_{3\eta}^{(s)} \to 0. \qquad (5.13b)$

Hence there is an automatic matching [equation (5.13b)] of the velocity gradient at the edge of the boundary layer to the potential flow at the surface. There may still be an order α^2 constant velocity, $\bar{u}_2$ = constant as $\eta \to -\infty$, allowed but we do not yet know this at the present stage.

Again, both for progressive [equations (5.8) and (5.10)] and for standing waves [equations (5.13)] the slow time variation enters only through the envelope function A.

6. The two-time-scale problem

We are doing an initial-value problem. The presence of viscosity will cause the wave field to decay with time. We can describe this wave attenuation through a two-time-scale analysis.

We introduce the two time scales

$$t = kc_0 t', \quad \tilde{t} = (2/R)\,t, \tag{6.1}$$

where t' is the dimensional time. The time derivative is transformed according to the chain rule:

$$\partial/\partial t \to \partial/\partial t + \Delta^2\,\partial/\partial \tilde{t}. \tag{6.2}$$

The aim is to develop asymptotic solutions for small Δ and to find the wave attenuation $A(\tilde{t})$. Thus we write in the surface layer

$$(u, w) = \alpha\{(\hat{u}_1, \hat{w}_1) + \Delta(u_2, w_2) + \Delta^2(u_3, w_3) + O(\Delta^3)\}, \tag{6.3a}$$

$$p = \alpha\{\hat{p}_1 + \Delta p_2 + \Delta^2 p_3 + O(\Delta^3)\}. \tag{6.3b}$$

At order α, we have potential flow, equations (3.1). In the surface boundary layer we rescale the normal co-ordinate and normal velocity on Δ:

$$\eta = (z-h)/\Delta, \quad w = \Delta W \quad \text{for} \quad R \to \infty. \tag{6.4}$$

At order $\alpha\Delta$, we have the boundary-layer solution

$$u_2 = A(\tilde{t})\,e^{\eta}\{\sin(x-\eta-t) - \cos(x-\eta-t)\}, \tag{6.5a}$$

$$W_2 = A(\tilde{t})\,\{\cos(x-t) - e^{\eta}\cos(x-\eta-t)\} + \partial h_1/\partial \tilde{t}, \tag{6.5b}$$

$$p_2 = 0. \tag{6.5c}$$

At order $\alpha^2\Delta^2$, the boundary-layer equations are

$$u_{3t} - \tfrac{1}{2}u_{3\eta\eta} = -p_{3x} - \hat{u}_{1\tilde{t}}, \tag{6.6a}$$

$$p_{3\eta} = 0, \tag{6.6b}$$

with the boundary conditions

$$u_{3\eta} = 0 \quad \text{on} \quad \eta = 0, \tag{6.6c}$$

$$u_3, W_3 \to 0 \quad \text{as} \quad \eta \to -\infty, \tag{6.6d}$$

$$p_3 = \hat{w}_{1z} \quad \text{on} \quad \eta = 0. \tag{6.6e}$$

Unless suppressed, the terms on the right-hand side of (6.6a) would give secular terms. If we use the normal-stress boundary condition (6.6e) to eliminate p_3 on the right-hand side of (6.6a), then the suppression of the right-hand side leads to the equation

$$\acute{A} + A = 0,$$

with the initial condition (3.5). Hence

$$A(\tilde{t}) = \exp(-\tilde{t}). \tag{6.7}$$

This result was found by Lamb (1932, p. 627) for *linear theory* of deep-water waves.

This temporal decay $\exp(-\gamma_T t')$ can be converted directly to spatial decay

(Phillips 1966, p. 145) $\exp(-\gamma_s x')$ through the *group velocity* c_g, which is given by

$$c_g = d(kc)/dk.$$

However, since $c \sim c_0 + O(\alpha^2)$ [see (2.2*d*)]

$$c_g = d(kc_0)/dk + O(\alpha^2) \tag{6.8}$$

(Wehausen & Laitone 1960). Hence the *dimensional spatial decay rate* γ_s is given by

$$\gamma_s = 2k^2\nu/c_g, \tag{6.9}$$

while the *dimensional temporal decay rate* γ_T is given by

$$\gamma_T = 2k^2\nu. \tag{6.10}$$

The above derivation focused on progressive waves but precisely the *same results apply to standing waves* as well.

7. Motion in the interior: progressive waves

Let us now review the evolution in our initial-value problem. First the pressure field establishes the potential flow immediately after the wave is imposed. Then the periodic motion of the fluid produces the boundary layers at both the bottom and the free surface. The flow field in the boundary layers is set up with a time scale of a wave period. From the moment of starting the waves, the vorticity, which is generated in the boundary layers, begins to diffuse into the fluid. In the interior of the fluid the motion will at first be irrotational, since no vorticity can be generated there. Let us investigate whether the vorticity will diffuse throughout the interior or whether it remains confined near the boundaries.

Let us examine the interior flow subject to the boundary conditions determined by the boundary layers. At the bottom, (4.5*b*) gives us

$$\overline{u}(-\surd\beta, \tilde{t}) = \tfrac{3}{4}\alpha^2 A^2(\tilde{t})/\sinh^2\surd\beta, \quad \overline{w}(-\surd\beta, \tilde{t}) = 0. \tag{7.1a}$$

At the top, (5.10) gives us

$$\overline{u}_z(0, \tilde{t}) = 2\alpha^2 A^2(\tilde{t})\coth\surd\beta, \quad \overline{w}_z(0, \tilde{t}) = 0. \tag{7.1b}$$

We can classify the quasi-steady interior flows by introducing the velocity scale $u_s = \alpha^2 c_0$ associated with the second-order drift. The intensity u_s of this drift is measured by the Reynolds number R_s (Stuart 1966), where

$$R_s = a^2k^3d^2c_0/2\nu = \beta(a/\delta)^2. \tag{7.2}$$

When $R_s \ll 1$, the convective nonlinearities of the Navier–Stokes equations are negligible and the interior circulation is a creeping flow. When $R_s \gg 1$, the convective nonlinearities are significant. These two limits correspond to $a \ll \delta$ and $a \gg \delta$ respectively.

If there is a quasi-steady interior flow $(\overline{u}, \overline{w})$ *explicitly* independent of t, then

$$\partial/\partial t = \Delta^2\, \partial/\partial\tilde{t}; \tag{7.3}$$

hence the local time derivative resulting from the wave attenuation is formally of the same order as the viscous forces.

The boundary conditions (7.1*a*, *b*) are consistent with a parallel flow solution $(\overline{u}(z,\tilde{t}),0)$ which satisfies, *independent of* R_s,

$$(\partial/\partial\tilde{t}-\tfrac{1}{2}\partial^2/\partial z^2)\,\overline{u} = -\overline{p}_x \tag{7.4}$$

and is given by

$$\begin{aligned}\overline{u}(z,\tilde{t}) = \alpha^2A^2(\tilde{t})\,\{&\coth\surd\beta\sin 2z+\tfrac{1}{2}\overline{p}_x\\ &+(3/4\sinh^2\surd\beta+\sin 2\surd\beta\coth\surd\beta-\tfrac{1}{2}\overline{p}_x)\cos 2z/\cos 2\surd\beta\}\end{aligned} \tag{7.5}$$

correct to $O(\alpha^2)$. This would be established on a dimensional time scale d^2/ν. Here the retention of the slow time variation $\partial/\partial\tilde{t}$ significantly modifies the field $\overline{u}$ compared with the 'conduction' solution of Longuet-Higgins (1953), which, if obtained in an Eulerian description, would satisfy

$$\tfrac{1}{2}\partial^2\overline{u}/\partial z^2 = \overline{p}_x. \tag{7.6}$$

The solution (7.5) is bounded for deep water, $\beta\to\infty$, whereas the solution of (7.6) is unbounded. There are, however, depths for which $\cos 2\surd\beta = 0$, i.e.

$$2kd = \tfrac{1}{2}n\pi,\quad n = 1,3,5,\ldots, \tag{7.7}$$

for which the solution (7.5) is unbounded. An examination of the initial-value problem (retaining $\partial/\partial t$ as well as $\partial/\partial\tilde{t}$) shows that at these depths *no quasi-steady state is attained*. In fact, $\overline{u}\sim A^2(\tilde{t})f(t)$, where $f(t)\sim t$ as $t\to\infty$. The slow envelope will ultimately cause the field to decay. When depths near these critical values are considered, very long times are required before a quasi-steady state is reached.

The solution (7.5), a modified version of the Longuet-Higgins conduction solution, exists for all R_s, apart from those critical depths given by (7.7). However, when R_s is large, this solution could well be unstable; other interior solutions would then exist.

8. Motion in the interior: standing waves

Let us examine the interior flow subject to the boundary conditions determined by the boundary layers. At the bottom, (4.7*b*) gives us

$$\left.\begin{aligned}\overline{u}(x,-\surd\beta,\tilde{t}) &= -\tfrac{3}{8}\alpha^2A^2(\tilde{t})\sin 2x/\sinh^2\surd\beta,\\ \overline{w}(x,-\surd\beta,\tilde{t}) &= 0.\end{aligned}\right\} \tag{8.1a}$$

At the top, (5.13*b*) gives us that

$$\overline{u}_z(x,0,\tilde{t}) = 0,\qquad \overline{w}_{\tilde{z}}(x,0,\tilde{t}) = 0. \tag{8.1b}$$

Again, the wave attenuation effects on the scale $\tilde{t}$ are comparable to the viscous forces. Boundary conditions (8.1) are not consistent with an exact parallel flow solution, however, and the convective nonlinearities of the Navier–Stokes equations will not vanish identically.

When $R_s \ll 1$, the governing equation is

$$\frac{\partial\overline{u}}{\partial t}-\frac{1}{2}\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right) = -\overline{p}_x. \tag{8.2a}$$

We find that the local time derivative exactly balances the x-vorticity diffusion and the profile of $\overline{u}$ is determined by

$$\tfrac{1}{2}\,\partial^2\overline{u}/\partial z^2 = \overline{p}_x. \tag{8.2b}$$

Since the total flux must be zero, a pressure gradient is set up,

$$\overline{p}_x = -\tfrac{9}{16}\alpha^2 A^2(\bar{t})\sin 2x/\sinh^2\sqrt{\beta}, \tag{8.2c}$$

and the resulting drift $\overline{u}$ is given by

$$\overline{u}(z,\bar{t}) = \alpha^2 A^2(\bar{t})\,\{\tfrac{9}{16}(\beta - z^2) - \tfrac{3}{8}\}\sin 2x/\sinh^2\sqrt{\beta} \tag{8.2d}$$

correct to second order in α. Hence the interior, which to $O(\alpha)$ is inviscid, is fully viscous to $O(\alpha^2)$.

When $R_s \gg 1$, the convective nonlinearities of the Navier–Stokes equations are significant and the vorticity produced in the boundary layers decays to zero within a layer of thickness δ^*, where

$$\Delta^* \equiv k\delta^* = \delta/a = \Delta/\alpha, \tag{8.3}$$

which is much thicker than the Stokes layer since $\alpha \ll 1$.

This thicker layer will be called the 'outer' layer, in contrast to the 'inner' layer, which is of Stokes type. Again, there exists the possibility of non-uniqueness of the solution and instability of the boundary-layer flow. However, we know of no evidence confirming that this does occur.

In order to obtain the structure of the outer layer on the bottom, we scale the normal co-ordinate and velocity component on Δ^*:

$$\zeta = (z+\sqrt{\beta})/\Delta^*, \quad \overline{W} = \Delta^*\overline{w} \quad \text{for} \quad R_s \to \infty.$$

The boundary-layer equations then become

$$\overline{u}\overline{u}_x + \overline{W}\overline{u}_z = \tfrac{1}{2}\overline{u}_{\zeta\zeta}, \tag{8.4a}$$

with

$$\overline{u}_x + \overline{W}_\zeta = 0, \tag{8.4b}$$

and the conditions (8.1) become

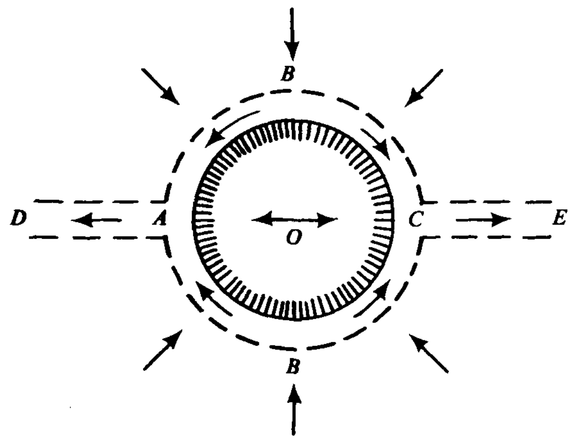
$$\overline{u} = -\tfrac{3}{8}\alpha^2 A^2(\bar{t})\sin 2x/\sinh^2\sqrt{\beta}, \quad \overline{W} = 0 \quad \text{on} \quad \zeta = 0. \tag{8.4c}$$

The matching conditions between the outer layer and the potential flow are

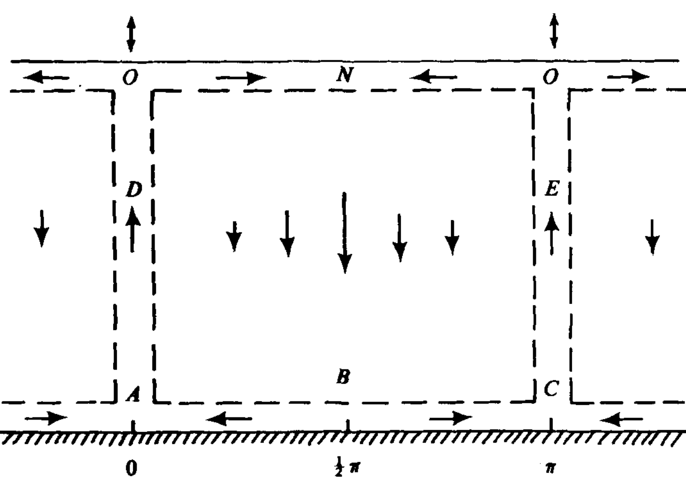
$$\overline{u}, \overline{W} \to 0 \quad \text{as} \quad \zeta \to \infty. \tag{8.4d}$$

Notice that, since $\partial/\partial z$ is large for $R_s \gg 1$, $\partial\overline{u}/\partial\bar{t}$ is negligible compared with the viscous forces.

The system (8.4) is equivalent to that governing the outer layer on a circular cylinder oscillating along its diameter, a problem that has been solved numerically by Davidson & Riley (1971). Figure 1 shows the analogy between the outer layer on the cylinder and the bottom outer layer in standing waves. The jet-like natures of the steady streaming are equivalent. The entrainment velocity $\overline{w}_\infty$ into the outer layer is seen to be negative for all values of x. Near the points $x = n\pi$, where n is an integer, jets emerge from the bottom boundary layer. The corresponding points on the cylinder are labelled equivalently, AO and CO are axes of oscillation. B is a node point with maximum entrainment velocity into the outer layer. At the points A and C, the outer layers from the two



(*a*)



(*b*)

FIGURE 1. The analogy between (*a*) the outer boundary layer on a cylinder oscillating along its diameter and (*b*) the outer boundaiy layer at the bottom of a water channel containing standing waves. In the latter, there are antinodes at points labelled O and a node at the point labelled N.

sides collide. After impact the outer-layer fluid moves in the directions AD and CE. The jets which emerge along the axis of oscillation will have the property that the momentum flux along them will be invariant; this may be seen from the work of Bickley (1937) and Davidson & Riley (1971). The surface outer layer is driven by the bottom outer layer owing to conservation of mass since *no drift-velocity gradient* is possible [equation (5.13*b*)]. Figure 2 shows sketches of the streamlines and the circulation of steady drift in standing waves having different Reynolds numbers R_s. Note the double cell structure in the vertical.

In summary then, when R_s is small, there is no outer layer. The vorticity will diffuse inwards from the boundary layers at the bottom and the free surface until a quasi-steady state, given by the viscous solution, is obtained. The interior motion contains

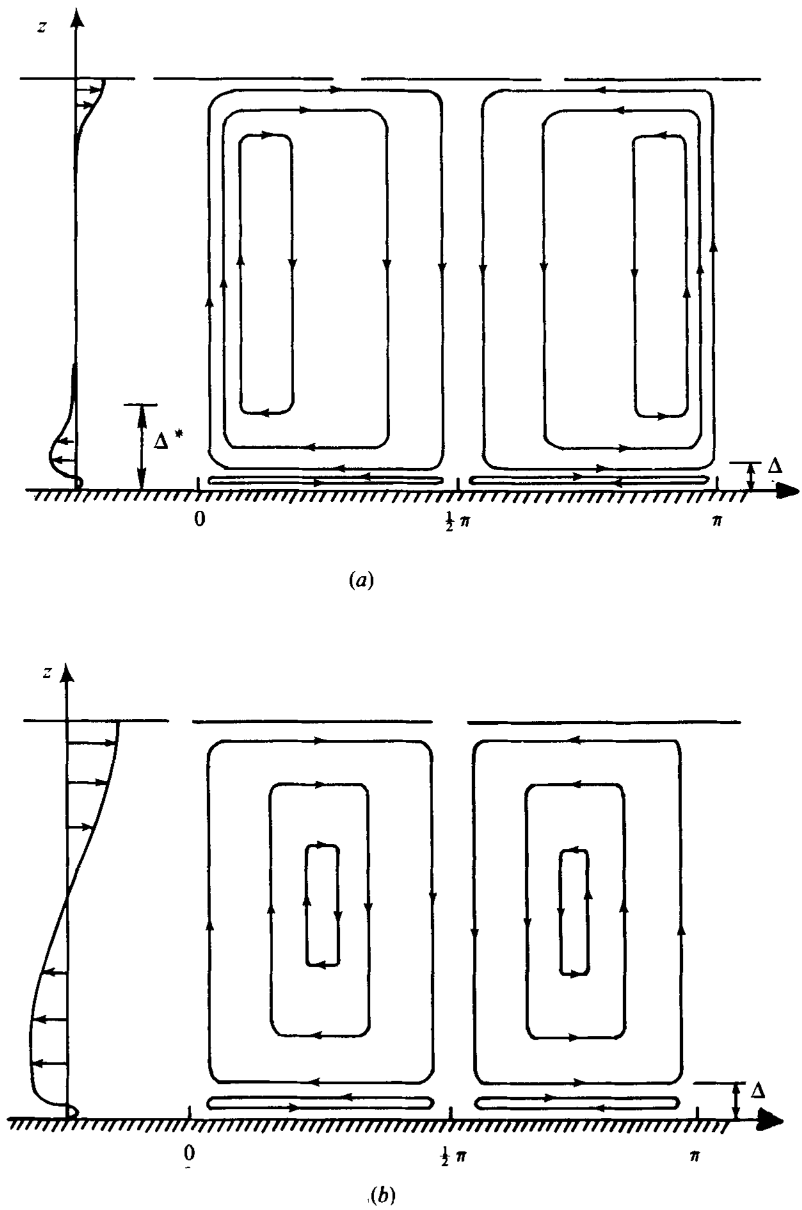


FIGURE 2. A sketch of the mean-drift streamlines for standing waves when (a) $R_s \gg 1$ and (b) $R_s \ll 1$. The profile of the x component of the drift is sketched in each case. The inner- and outer-layer thicknesses are indicated by Δ and Δ^* respectively. At the surfaces there are antinodes at $x = 0, \pi$ and nodes at $x = \frac{1}{2}\pi$.

a second-order vorticity determined by the oscillatory layers at the boundaries. The time taken for the vorticity to diffuse into the interior and for a quasi-steady state to be reached will be of the order of d^2/ν. When R_s is large, there are two outer layers within which the steady drift decays to the potential flow. Outside the outer layers there will be an inviscid flow. The drift motion of the fluid forms a cellular structure. The whole wave field is exponentially decaying with the slow time $\tilde{t}$.

9. Mass-transport velocity

The previous analysis has been in Eulerian variables, but if we are interested in the motion of individual fluid elements, a Lagrangian description is required. The Lagrangian properties of the motion are derivable from the Eulerian solutions already found (Longuet-Higgins 1953).

Let $\mathbf{U}(\mathbf{x}_0, t)$ denote the velocity of the particle whose co-ordinates at time $t = 0$ are $\mathbf{x}_0$. Then we have

$$\mathbf{U} = \mathbf{u}\left(\mathbf{x}_0 + \int_0^t \mathbf{U} dt, t\right) = \mathbf{u}(\mathbf{x}_0, t) + \int_0^t \mathbf{U} dt \cdot \text{grad}\, \mathbf{u}(\mathbf{x}_0, t) + O(\alpha^3) \tag{9.1}$$

by a Taylor series expansion. Since $\mathbf{U}$ is of the same order as $\mathbf{u}$, we assume that

$$\mathbf{U} = \alpha \mathbf{U}_1 + \alpha^2 \mathbf{U}_2 + O(\alpha^3), \tag{9.2}$$

whence, on substituting into (9.1) and equating to zero coefficients of like powers of α, we have

$$\overline{\mathbf{U}}_1 = \bar{\mathbf{u}}_1, \quad \overline{\mathbf{U}}_2 = \bar{\mathbf{u}}_2 + \overline{\int_0^t \mathbf{u}_1 dt . \,\text{grad}\, \mathbf{u}_1}, \tag{9.3a, b}$$

where the bar denotes the mean value with respect to time over a complete period with $\mathbf{x}_0$ fixed. The first-order motion is periodic in time, so we have

$$\overline{\mathbf{U}}_1 = \bar{\mathbf{u}}_1 = 0. \tag{9.4}$$

The components of the mass-transport velocity are defined as follows:

$$\overline{U}_2 = \bar{u}_2 + \overline{[\textstyle\int u_1 dt]\, u_{1x}} + \overline{[\textstyle\int w_1 dt]\, u_{1z}}, \tag{9.5a}$$

$$\overline{W}_2 = \bar{w}_2 + \overline{[\textstyle\int u_1 dt]\, w_{1x}} + \overline{[\textstyle\int w_1 dt]\, w_{1z}}. \tag{9.5b}$$

In the bottom boundary layer, the Eulerian velocity (4.5*a*) for *progressive waves* becomes through (9.5*a*)

$$\overline{U}_2 = (A^2(\bar{t})/\sinh^2 \surd\beta)\{\tfrac{5}{4} - 2e^{-\eta} \cos \eta + \tfrac{3}{4} e^{-2\eta}\}. \tag{9.6a}$$

This is plotted in figure 3(*a*). When $\eta \to \infty$, this mass-transport velocity becomes

$$\overline{U}_2 \to \tfrac{5}{4} A^2(\bar{t})/\sinh \surd\beta + O(\alpha^3). \tag{9.6b}$$

In the bottom boundary layer, the Eulerian velocity (4.7*a*) for *standing waves* becomes through (9.5*a*)

$$\overline{U}_2 = (A^2(\bar{t}) \sin 2x/8 \sinh^2 \surd\beta)\{-3 + 3e^{-2\eta} + 8e^{-\eta} \sin \eta\}. \tag{9.7a}$$

This is plotted in figure 3(*b*). When $\eta \to \infty$, this mass-transport velocity becomes

$$\overline{U}_2 \to -\tfrac{3}{8} A^2(\bar{t}) \sin 2x/\sinh^2 \surd\beta + O(d^3). \tag{9.7b}$$

Apart from the attenuation envelope results (9.6*b*) and (9.7*b*) are identical with those obtained by Longuet-Higgins (1953).

In the surface boundary layer, the mass-transport velocity is defined by (Longuet-Higgins 1953)

$$\overline{U}_2 = \bar{\dot{s}}_2 + \overline{\left[\int \dot{s}\, dt\right] \frac{\partial \dot{s}}{\partial s}} + \overline{\left[\int \dot{n}\, dt\right] \frac{\partial \dot{s}}{\partial n}}, \tag{9.8a}$$

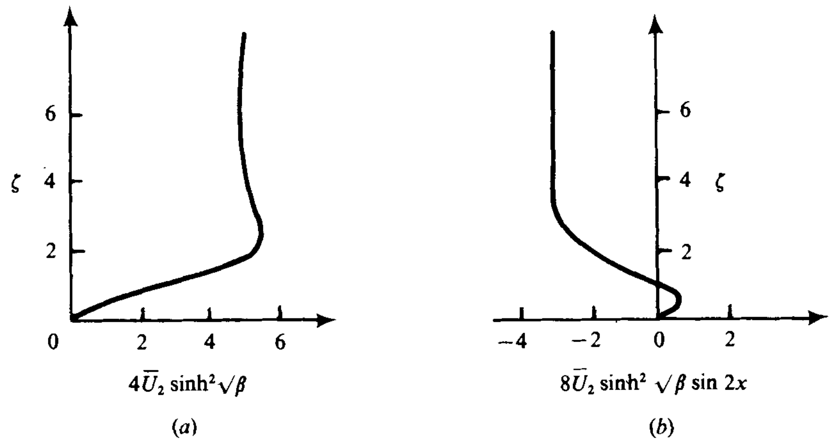


FIGURE 3. Sketches of the profiles of mass-transport velocity in the bottom boundary layer for (a) progressive waves and (b) standing waves.

and the vertical gradient of the mass-transport velocity can be found by differentiation:

$$\frac{\partial \bar{U}_2}{\partial n} = \overline{\frac{\partial \dot{s}_2}{\partial n}} + 2\overline{\left[\int \frac{\partial \dot{s}}{\partial n} dt\right]\frac{\partial \dot{s}}{\partial s}} + \overline{\left[\int \dot{s}\, dt\right]\frac{\partial^2 \dot{s}}{\partial s\, \partial n}} + \overline{\left[\int \dot{n}\, dt\right]\frac{\partial^2 \dot{s}}{\partial n^2}}, \tag{9.8b}$$

where $(\dot{s}, \dot{n})$ denotes the rate at which the co-ordinates of a particular element of fluid are increasing with time.

For *progressive waves*, the Eulerian velocity (5.8*a*) relative to the surface boundary becomes through (9.8*b*)

$$\partial \bar{U}_2/\partial n = 4A^2(\bar{t}) \coth \sqrt{\beta}\{1 - e^{\eta} \cos \eta\}. \tag{9.9a}$$

When $\eta \to -\infty$, this mass-transport gradient becomes

$$\partial \bar{U}_2/\partial n \to 4A^2(\bar{t}) \coth \sqrt{\beta}. \tag{9.9b}$$

For *standing waves*, the Eulerian velocity gradient in the surface boundary layer becomes through (9.8*b*)

$$\partial \bar{U}_2/\partial n = -2A^2(\bar{t}) \coth \sqrt{\beta}\, e^{\eta} \sin \eta \sin 2x. \tag{9.10}$$

When $\eta \to -\infty$, the mass-transport velocity gradient is zero.

Apart from the attenuation envelope these results are identical with those obtained by Longuet-Higgins (1953).

In the interior *for progressive waves*, we have already found the velocity field, (3.1*b*, *c*) and (7.5). It follows from (9.5*a*) that

$$\bar{U}_2 = A^2(\bar{t})\left\{\frac{\coth 2(z+\sqrt{\beta})}{2\sinh^2\sqrt{\beta}} + \coth\sqrt{\beta} \sin 2z + \left(\frac{3}{4\cos 2\sqrt{\beta}\,\sinh^2\sqrt{\beta}} + \tan 2\sqrt{\beta} \coth\sqrt{\beta} - \frac{\bar{p}_x}{2\cos 2\sqrt{\beta}}\right)\cos 2z + \frac{\bar{p}_x}{2}\right\}. \tag{9.11a}$$

Longuet-Higgins (1953) wished to apply his results to waves in a closed tank. He attempted to simulate this effect by imposing a pressure gradient $\bar{p}_x$ so that the volume

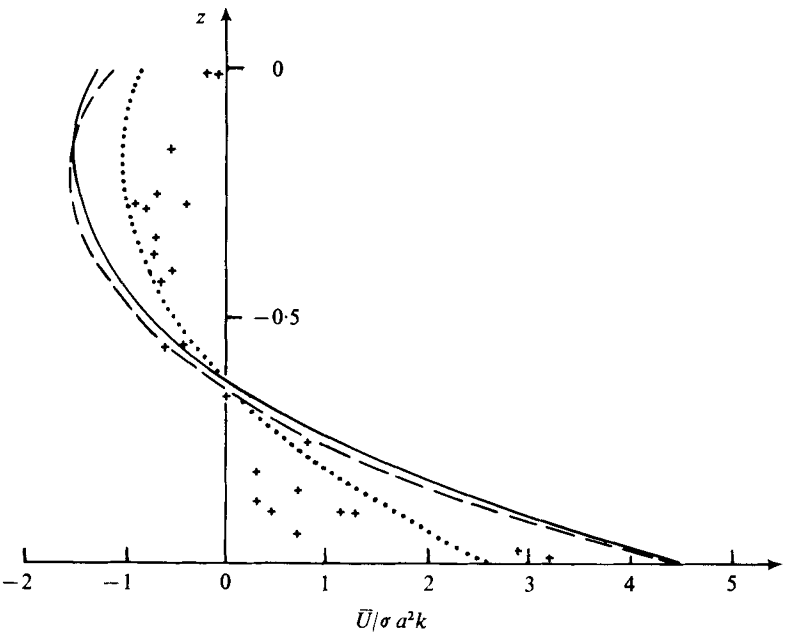


FIGURE 4. The profiles of mass-transport velocity in the interior in progressive waves for $\sqrt{\beta} = 0{\cdot}50$. — — — —, Longuet-Higgins (1953) conduction solution; ——, present analysis for $\tilde{l} = 0$; ······, present analysis for $\tilde{l} = \beta = 0{\cdot}25$; +, data of Russell & Osorio (1957) for $a = 11{\cdot}7$ cm, $\omega = 4{\cdot}2$ rad/s, $d = 50{\cdot}8$ cm.

flow rate across any vertical section would be zero. This induced pressure gradient is given by

$$\bar{p}_x = -\frac{2A^2(\tilde{l})}{2\sqrt{\beta}-\tan 2\sqrt{\beta}}\left\{\coth\sqrt{\beta}\cos 2\sqrt{\beta}+\frac{3\tan 2\sqrt{\beta}}{4\sinh^2\sqrt{\beta}}+\tan^2\sqrt{\beta}\coth\sqrt{\beta}\sin 2\sqrt{\beta}\right\}. \tag{9.11b}$$

At the surface, the mass-transport velocity is then found to be

$$\bar{U}_2(0) = A^2(\tilde{l})\left\{\frac{\cosh 2\sqrt{\beta}}{2\sinh^2\sqrt{\beta}}+\frac{3}{4\cos 2\sqrt{\beta}\sinh^2\sqrt{\beta}}+\tan^2\sqrt{\beta}\coth\sqrt{\beta}+\frac{\bar{p}_x}{2}\left(1-\frac{1}{\cos 2\sqrt{\beta}}\right)\right\}. \tag{9.11c}$$

However, for deep water, $\beta \gg 1$, it can be seen that $\bar{p}_x$ can no longer be maintained constant. This reflects the fact that the convection terms in the momentum equations are important and hence the return flow can be defined only if the conditions at $x \to \pm\infty$ are specified.

The diffusion time scale for the vorticity to diffuse into the whole interior and for the quasi-steady state to be reached will be $t' = O(d^2/\nu)$. Figures 4 and 5 show our parallel flow profiles of interior mass-transport velocity for progressive waves having respectively $\sqrt{\beta} = 0{\cdot}50$ and $1{\cdot}25$. The dashed curves are those given by the Longuet-Higgins (1953) conduction solution while the solid curves are those of the present analysis with $\tilde{l} = 0$. Notice that the z structures differ from those of Longuet-Higgins, the discrepancy becoming greater for larger depths d, i.e. larger β. However, neither set of curves gives

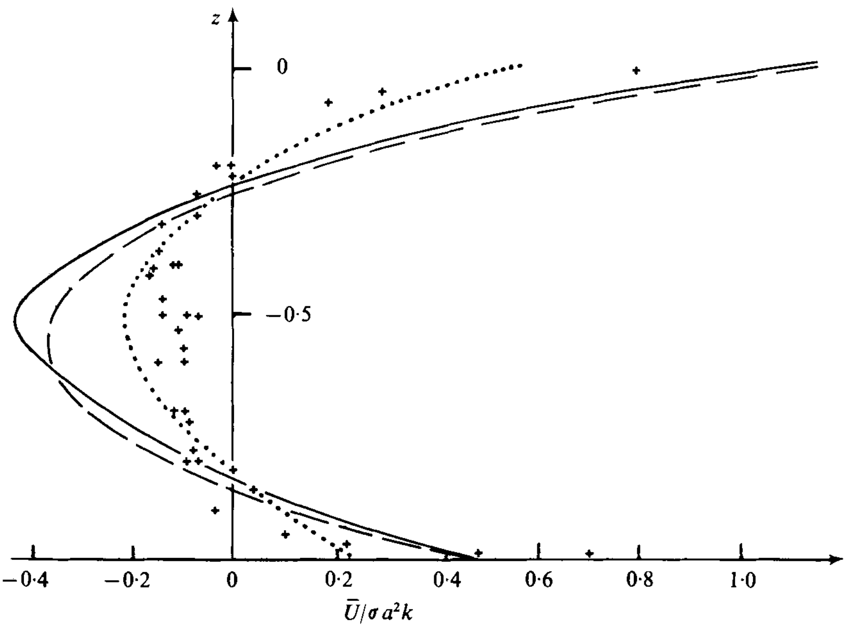


FIGURE 5. The profiles of mass-transport velocity in the interior in progressive waves for $\surd\beta = 1\cdot25$. ---, Longuet-Higgins (1953) conduction solution; ——, present analysis for $\bar{t} = 0$; ······, present analysis for $\bar{t} = \frac{1}{4}\beta = 0\cdot39$; +, data of Russell & Osorio (1957) for $a = 11\cdot7$ cm, $\omega = 2\cdot1$ rad/s, $d = 50\cdot8$ cm.

remarkable agreement with the plotted experimental results of Russell & Osorio (1957). The direct formal conversion of our results to those having spatial decay (§ 6) results in the following estimates. If $d = 50$ cm, $k = 0\cdot01\,\text{cm}^{-1}$ ($\surd\beta = 0\cdot50$), $a = 5$ cm, $c_0 = 80$ cm/s, $c_g = 80$ cm/s and $\nu = 0\cdot01\,\text{cm}^2/\text{s}$, then from (6.10*b*)

$$\gamma_T = 2 \times 10^{-6}\,\text{s}^{-1}, \tag{9.12a}$$

and from (6.10*a*)

$$\gamma_s = 2\cdot5 \times 10^{-8}\,\text{cm}^{-1}. \tag{9.12b}$$

Russell & Osorio (1957) found that varying their measurement station from 40 ft to 140 ft downstream from the wave maker resulted in a small deviation in their drift data. This is borne out by evaluating $\exp(-\gamma_s \Delta x')$ for $\Delta x' = 100$ ft: $\exp(-\gamma_s \Delta x') \approx 0\cdot9993$. They also obtained a deviation in their data by waiting for times comparable to the diffusion time. The decay (in time) after $\Delta t' = d^2/\nu$ assuming that there is *no wave maker* (external energy supply) would be given by $\exp(-\gamma_T \Delta t') = \exp(-\gamma_T d^2/\nu = \exp(-2\beta)$ so that for $\surd\beta = 0\cdot50$ and $1\cdot25$, $\exp(-\gamma_T \Delta t')$ decays to $0\cdot606$ and $0\cdot044$ respectively. Since the mass transport is proportional to $[\exp(-\gamma_T \Delta t')]^2$ this decay would, indeed, be substantial. The dotted curves on figures 4 and 5 correspond to $t' = 0\cdot50\, d^2/\nu$ and $0\cdot12\, d^2/\nu$ respectively and are *included only as an indicator* of the types of profiles to be expected when no energy source exists. *We do not claim that the seeming improvement of the agreement is necessarily real* since the experiment and analysis do not apply to identical systems. At best our rough comparisons may be applicable away from a wave maker, in whose neighbourhood very complicated initial values (in space) would be appropriate. The sentiments of Russell & Osorio (1957) still seem valid for progressive waves: "For the surface and interior of the fluid there is no strictly applicable theory."

For *standing waves*, the mass-transport velocity field in the interior region is the same as the Eulerian velocity field since the second and third terms on the right-hand side of (9.5*a*) are identically zero. A sketch of the streamlines and the circulation of mass transport in a standing wave is shown in figure 2.

10. Mean vorticity just beneath the surface boundary layer

For progressive waves, it was shown by Phillips (1966), that, in an Eulerian description of the motion, the wave momentum is contained in the region of space above the wave troughs. Hence a decrease in the mean momentum of this region must be accompanied by a mean stress across horizontal planes below the free surface, which is clearly of the second order. In Cartesian co-ordinates (x, z, t) a vertical integration of the horizontal momentum equation from a level $z = -z_0$ just beneath the surface boundary layer at the wave troughs to the free surface (figure 6) followed by a time average over one cycle leads to

$$\overline{u_t h} - \overline{[uw]_{-z_0}} = \tfrac{1}{2}\Delta^2 \overline{[u_{zz}]^h_{-z_0}}. \tag{10.1}$$

Owing to the wave attenuation (7.1*c*) and the $O(\alpha)$ potential solution (3.1), the first term is then

$$\overline{u_t h} = \Delta^2 \overline{u_{\tilde{t}} h} = -\Delta^2 \overline{u_1 h_1} = -\tfrac{1}{2}\alpha^2\Delta^2 A(\tilde{t}) \coth\sqrt{\beta} + O(\alpha^3). \tag{10.2}$$

In Cartesian co-ordinates, owing to the wave decay, we found from (6.5*b*) and (3.1*a*) that

$$W_2 = -A(\tilde{t})\, e^{\eta} \cos(x - \eta - t).$$

The Reynolds stress can then be found from (6.5*a*, *b*) and (3.1) to be

$$\overline{uw} = -\tfrac{1}{2}\alpha^2\Delta^2 A^2(\tilde{t}) \coth\sqrt{\beta}\{-\eta\, e^{-\eta} \sin\eta + (\eta - 1)\, e^{\eta} \cos\eta\}. \tag{10.3}$$

At the level $z = -z_0$, which is outside the surface boundary layer $\eta \to -\infty$, the Reynolds stress is given by

$$-\overline{[uw]_{-z_0}} = 0, \tag{10.4}$$

and there is no contribution to this order from the potential region, the shaded region of figure 6. Furthermore, since the surface condition (2.1*f*) of zero shear stress is in curvilinear co-ordinates, we must transform co-ordinates (5.9) and convert to Cartesian velocity components to obtain

$$\overline{(\partial u/\partial z)_h} = \alpha^2 A^2(\tilde{t}) \coth\sqrt{\beta}. \tag{10.5}$$

If (10.2), (10.4) and (10.5) are substituted into (10.1), we obtain

$$[\partial\bar{u}/\partial z]_{-z_0} = 2\alpha^2 A^2(\tilde{t}) \coth\sqrt{\beta}, \tag{10.6}$$

which we can write as

$$[\partial\bar{u}/\partial z]_{z=0} = 2\alpha^2 A^2(\tilde{t}) \coth\sqrt{\beta}, \tag{10.7}$$

correct to the second order in α. This is the mean velocity gradient (mean vorticity) just outside the surface boundary layer and this result agrees with the matching boundary condition for the interior problem in Cartesian co-ordinates (5.10). We see from (6.5*b*) that the $O(\alpha\Delta)$ vertical flow vanishes outside the surface boundary layer, so that it cannot produce an $O(\alpha^2\Delta^2)$ Reynolds stress there. Hence an $O(\alpha^2)$ mean

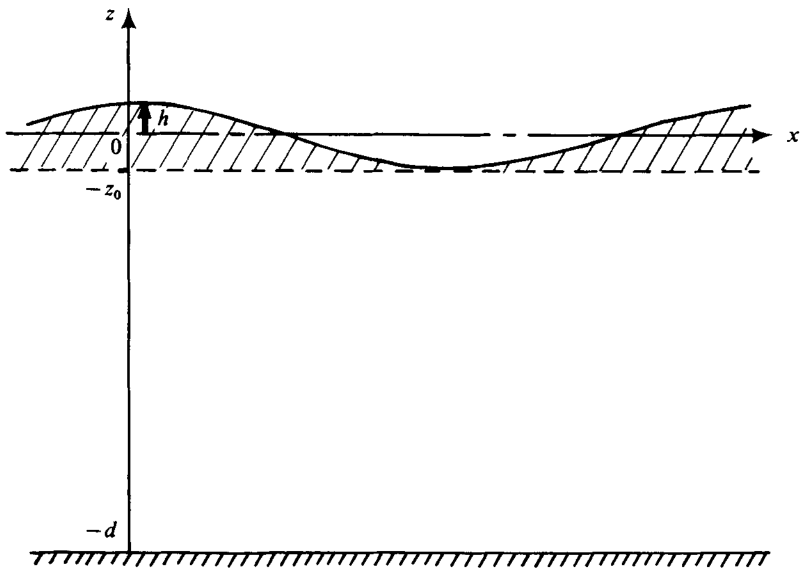


FIGURE 6. The region over which the momentum equation is integrated.

viscous stress is set up as shown in (10.1) on $z = -z_0$ which balances two numerically equal effects: half the momentum loss $\overline{u_t h}$ from (10.2) and the free shear stress (10.5). This is in contrast to the balance between viscous stresses and total momentum loss suggested by Phillips (1966, p. 36) although the final results are the same.

The mean vorticity $\overline{\Omega}$ at the outer edge of the surface boundary layer can be found from (s, n, t) co-ordinates (5.8b) and is given by

$$\overline{\Omega} = \overline{u}_n + \overline{\kappa u} = 2\alpha^2 A^2(\tilde{t}) \coth \sqrt{\beta}, \qquad (10.8)$$

which agrees with the result which Dore (1971) obtained using a (different) set of orthogonal curvilinear co-ordinates.

11. Discussions and conclusions

The present analysis considers an initial-value problem for slightly viscous water waves. It is seen that local viscous attenuation in time (of scale $\tilde{t}$) can be included in the analysis by using the method of multiple scales. The attenuation does not dramatically alter the boundary-layer structure but it does significantly modify the interior motions, where its effect is comparable with that of the boundary-layer corrections.

For *progressive waves* the boundary layers at the bottom and at the free surface create vorticity that diffuses inwards and determines an interior flow. There an interior flow solution is a superposition of an $O(\alpha)$ potential flow plus an $O(\alpha^2)$ balance between local attenuation and viscous forces which is established on the time scale d^3/ν for vertical diffusion. The Lagrangian version of this interior state is related to the 'conduction' solution of Longuet-Higgins but since the local attenuation cannot be neglected, it never coincides with it, the departure being most pronounced for large values of d. This drift is bounded for $d \to \infty$, again in contrast to the Longuet-Higgins solution. However, at certain critical depths secular behaviour rather than quasi-steady behaviour is predicted This solution exists independent of the size of the ratio δ/a and so can more logically be compared with the experiments of Russell & Osorio (1957). Figures 4 and 5 make these comparisons. Care, however, is necessary in this comparison since time decay in the analysis is equated to spatial decay in the experiment. This is especially true near a wave maker, where a harmonically 'pure' initial condition (in space) would probably be an artificial imposition. Sufficiently far downstream, however, the comparison might be reasonable.

The strong forward velocities near the bottom which are observed in experiment are accounted for quantitatively by the theory. We may expect a forward bending of the velocity profile near the free surface for progressive waves, but no careful observations are yet available because of the experimental difficulty in making measurements close to a moving surface and the weak stability of the motion near the surface (Longuet-Higgins 1960).

For *standing waves* the boundary layers at the bottom and at the free surface create vorticity that diffuses inwards. The introduction of Stuart's (1966) Reynolds number R_s characteristic of the steady drift allows us to classify the cases of interior motion as viscous or inviscid. When $R_s \gg 1$, there is a double-boundary-layer structure both on the bottom and on the free surface. The thinner (inner) layers are Stokes layers that respectively produce drift and drift gradients on the bottom and top. The thicker (outer) layers balance viscous forces with convection and take the drift profiles of the thinner layers to the potential flow in the core. The slow decay here enters only through the decaying envelope. When $R_s \ll 1$, the thicker layers merge, so that the interior flow is fully viscous. Here downstream vorticity diffusion balances the slow local acceleration. In both cases the drift forms the closed cellular structure depicted in figure 2. There is an equivalence between the outer layer at the channel bottom and the outer layer found on a circular cylinder oscillating along its diameter (figure 1). The outer layer at the surface is driven by that at the bottom. These predictions concerning the mass-transport velocity occurring in closed cellular patterns seem capable of an experimental test, both for their existence and for their range of applicability.

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